Shock Diffusion in Regular Networks: The Role of Transitive Cycles

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Shock diffusion in large regular networks: the role of transitive cycles*

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Abstract

We study how the presence of transitive cycles in the interbank network affects the extent of financial contagion. In a regular network setting, where the same pattern of links repeats for each node, we allow an external shock to propagate losses through the system of linkages (interbank network). The extent of contagion (contagiousness) of the network is measured by the limit of the losses when the initial shock is diffused into an infinitely large network. This measure indicates how a network may or may not facilitate shock diffusion in spite of other external factors. Our analysis highlights two main results. First, contagiousness decreases as the length of the minimal transitive cycle increases, keeping the degree of connectivity (density) constant. Second, as density increases the extent of contagion can decrease or increase, because the addition of new links might decrease the length of the minimal transitive cycle. Our results provide new insights to better understand systemic risk and could be used to build complementary indicators for financial regulation.

JEL Classification: G33, D85, C69, C02
Keywords: Financial contagion, networks, shock diffusion, transitive cycles, degree

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1 Introduction

Financial contagion through counterparty risk is commonly accepted to be one of the hallmarks of the global financial crisis that started in 2007. Since the pioneering works by Allen and Gale (2000) and Freixas et al. (2000), many studies have analyzed how the structure of financial networks affects the propagation of an external shock (see Allen and Babus 2009, Summer 2013, Cabrales et al 2016, Hser 2015, or Glasserman and Young 2016 for reviews of this stream of literature.) The literature has uncovered the role played by certain characteristics of the network, focusing notably on density, which relates to the average number of neighbors or average degree in the network.[1] With different methodologies, this stream of literature shows that the effect of network density on shock diffusion is non-monotonic and depends on factors as the size of the shock, the presence of financial acceleration, level of integration, or the diversification of the system as a whole.[2]

Nevertheless, little is known about the effect of other characteristics of the network with the exceptions of Craig et al. (2014) and Rogers and Veraart (2013) on individual centrality, or Allen et al. (2012) on clustering. We contribute to this literature by studying the role of transitive cycles in facilitating or restraining the propagation of a shock in financial networks. Our model shows that the length of transitive cycles is an important factor that shapes the relationship between network density and shock diffusion.

To lay out the intuitive foundation, consider two different structures of financial interdependencies as depicted in Figure 1. We will provide formal definitions in the next section. An arrow from bank 1 to bank 2 indicates that bank 2 will take a loss if bank 1 fails. We call bank 1 an in-neighbor of bank 2 and bank 2 an out-neighbor of bank 1. In both networks (a) and (b) represented below, each institution has two in-neighbors and two out-neighbors. Nevertheless, these two networks are not identical, or isomorphic, due to the different structure of *cycles* they each possess.

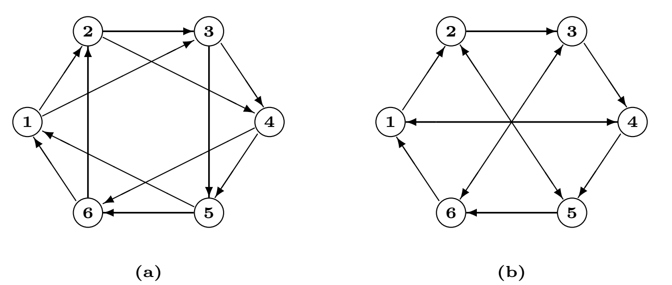


Figure 1: Same degree, different cycle length

We observe cycles of different length for each structure. In network (a) 1 can affect

[1] Acemoglu et al 2015, Battiston et al 2012, Blume et al 2011, 2013, Elliot et al 2014, Gai et al 2011, Nier et al 2007, Gai and Kapadia 2010, Haldane and May 2011, Gofman 2014, Acharya 2009, Cabrales et al 2013, Wagner 2011, Ibragimov, Jafee and Walden 2011, Castiglionesi and Eboli 2017, among others.

[2] A higher density implies higher *individual* diversification but it does not necessarily mean more *systemic* diversity.

2, 2 can affect 3, and 1 can affect 3. We call this transitivity of loss-given-default among financial institutions a *transitive cycle*. In network (b) 1 can affect 2, 2 can affect 3, 3 can affect 4, and 1 can affect 4. In network (b) the transitive cycles always include at least four banks, while in network (a) they only include three banks. Therefore, the length of the minimal transitive cycle is smaller in network (a) than in network (b).

We model the structure of financial liabilities as a directed network. When a bank defaults after taking a large external shock, it will impose losses on other banks to which it has liabilities. The losses-given-default in turn may cause these banks to fail. Thus, losses propagate into the network as a flow through a system of linkages. Inspired by Morris (2000), we assume that the population is infinite but each bank has a finite number of links, in our case with an identical pattern.[3] This type of structure is what we consider a large regular network. In this setting, we measure how a structure facilitates shock diffusion by computing the limit of the individual loss when the distance between a bank and the initial shock goes to infinity. A small value of this measure indicates that the structure itself is robust and can restrain the diffusion of the initial shock to a long distance. We therefore take this measure as an indicator of the contagiousness of the network.

In our setting, we show that the contagiousness of the network decreases as the length of the minimal transitive cycle increases, while keeping the number of links equal and constant for all nodes. Furthermore, increasing the connectivity of the network (or level of diversification of the liabilities structure) can have ambiguous effects on contagiousness. This ambiguity arises because when connectivity increases additional links may or may not decrease the length of the minimal transitive cycle. On the one hand, when additional links do not change the length of minimal transitive cycle (long links are added), contagiousness decreases as connectivity increases. On the other hand, when additional links are made to banks at a closer distance than the length of minimal transitive cycle (short links are added), the length of the minimal transitive cycles decreases. In this case, contagiousness decreases as connectivity increases if and only if the length of the minimal transitive cycle is above a certain threshold. If the length of the minimal transitive cycle is lower than the threshold, increasing connectivity by adding links to banks that are relatively close will result in an increase of contagiousness.

To extend our analysis, we study the contagiousness of regular networks versus different structures having some related characteristics. First, we compare regular networks to tree networks with same out-degree. The contagiousness of the tree networks always tends to zero as long as the out-degree of each node is greater than one. We note that the contagiousness of regular network approaches the one of tree networks as the length of the minimal transitive cycles approaches infinity. We next use complete multipartite networks

[3]The assumption of an infinite population allows us to draw more general conclusions about the effect of the length of the minimal transitive cycle. If each bank has assets and liabilities to a finite number of other banks, and the total number of banks is finite, a few values of length of minimal transitive cycle are compatible. By allowing the total number of banks to be large enough we also allow for the length of the minimal transitive cycle to go from 3 to infinity.

as a benchmark for comparison. Complete multipartite networks have the property of keeping the losses constant as the initial shock diffuses into the system. This constant loss is equal to the reduction in asset value of the direct neighbors of the first defaulted bank. Again we find a threshold for the length of the minimal transitive cycles, above which the contagiousness of regular structures is smaller than the one of the multipartite networks.

These results suggest some guidelines for policy makers. First, many systemic risk indicators have been developed, with several ones that take into account the structure of the financial system together with financial acceleration (for example, DebtRank by Battiston et al. 2012b, or Contagion Index by Cont et al. 2012). Our measure, focusing solely on the structure of the network, could be used to build complementary indicators. Knowing which region has high potential for shock diffusion may help regulators to devise appropriate interventions in time of crisis. Furthermore, as the measure is derived without complex financial mechanisms, its application can be adapted to other type of financial interdependencies, such as networks of payments.

Second, the Basel Committee on Banking Supervision has compiled a set of global standards for financial institutions since 1982. One of the most important objectives is to improve the banking sector's ability to absorb shocks arising from financial and economic stress. In response to the 2007 global financial crisis, Basel III specifies extra recommendations for systemically important financial institutions (SIFI). Going one step further, the European Commission has decided to transpose some of the Basel III recommendations into laws that will be enforced starting in 2019 for the European Union. These recommendations focus mainly on variables at the individual level such as capital requirement, liquidity, and leverage ratio, with surcharge to SIFIs due to their potential important impact to the financial system. In what concerns the results presented in this paper, it would be useful to have complementary regulations on the structure itself of the financial linkages. Banks have to be more careful when choosing their diversification strategies, as increasing the level of diversification might facilitate the diffusion of potential shocks, especially when the length of the minimal transitive cycle decreases.

This paper is organized as follows. We introduce the setting in Section 2. The results are stated in Section 3. We provide a discussion of our results in Section 4 and conclude in Section 5.

2 The model

2.1 The financial interdependencies

In this section we introduce the basic notions and definitions that are needed in the subsequent analysis. The interested reader can check the initial chapters of recent books covering network-related definitions and measures more exhaustively. To name but a few, see Goyal (2007), Jackson (2008), or Newman (2010).

Let $N = \{1, 2, ..., n\}$ denote the set of financial institutions or banks. Each bank $i \in N$

holds a capital buffer $w_i \geq 0$, owns external assets for an amount of $a_i \geq 0$, and has liabilities to other banks $l_{ij} \geq 0$, where $j \in N$, $j \neq i$. The total interbank liability held by bank i is given by $L_i = \sum_j l_{ij}$. Bank i's total assets are therefore given by $a_i + \sum_k l_{ki}$ and banks i's total liabilities are given by $w_i + \sum_j l_{ij}$.

This interdependence can be represented by a (directed) graph over N where the set of links g is defined by $ij \in g$ for $i \in N$ and $j \in N$ if and only if $l_{ij} > 0$. To keep the model tractable, we have taken some regularity assumptions regarding the financial interdependence network.

Given a bank i, we define i's out-neighborhood to be the set of banks to whom i has a liability, i.e., $N_i^{out}(g) = \{j \in N \text{ such that } l_{ij} > 0\}$. The cardinality of i's out-neighborhood is called i's out-degree and denoted by k_i^{out}. Similarly, let i's in-neighborhood be the set of banks that have a liability with i, i.e., $N_i^{in}(g) = \{j \in N \text{ such that } l_{ji} > 0\}$. The cardinality of i's in-neighborhood is called i's in-degree and denoted by k_i^{in}.

A path in the network (N, g) is a set of consecutive links $\{i_1i_2,\ i_2i_3,\ ...,\ i_{r-1}i_r\} \subseteq g$ with $i_s \in N$ for all $s = 1, .., r$ and $i_si_{s+1} \in g$ for all $s = 1, .., r-1$. The length of a path is the number of links in it. We say that j is connected to i if there is a path $\{i_1i_2,\ i_2i_3,\ ...,\ i_{r-1}i_r\} \subseteq g$, such that $i_1 = i$ and $i_r = j$. The distance between i and j in the network (N, g), denoted $d(i, j)$, is the number of links in the shortest path that connects i to j or vice versa (the path with smallest distance between two players is called a geodesic). A subset of nodes $S \subseteq N$ is connected in the network (N, g) if for every pair of nodes i and j in S either i is connected to j or j is connected to i. The network (N, g) is connected if N is connected in (N, g). We denote by $N_i^{out,\infty}$ the set of nodes that are connected to i in (N, g) and by $N_i^{in,\infty}$ the set of nodes to whom i is connected in (N, g).

A *transitive cycle* in the network is a path such that there exists *distinct* nodes $\{i_1, ..., 1_c\} \subseteq N$ satisfying that $\{i_1i_2,\ i_2i_3,\ ...,\ i_{c-1}i_c,\ i_1i_c\} \subseteq g$. An intransitive cycle in the network is a path such that there exists *distinct* nodes $\{i_1, ..., 1_c\} \subseteq N$ satisfying that $\{i_1i_2,\ i_2i_3,\ ...,\ i_{c-1}i_c,\ i_ci_1\} \subseteq g$. Note that our cycles are "minimally" defined because in our definition the nodes in the cycle are distinct (a node cannot be visited several times). The length of a cycle is the number of links in the cycle, which by our definition of a cycle is also equal to the number of participants in the cycle. Figure 2 below shows a transitive and an intransitive cycle of length $c = 4$.

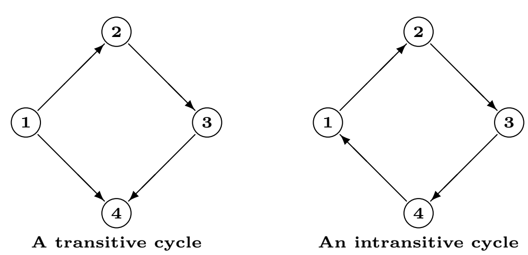


Figure 2: Cycles of length 4

To keep the model tractable, we make some regularity assumptions regarding the structure of the network. A financial network is homogeneous if all banks have the same and equal out-degree and in-degree, i.e. $k_i^{in} = k_i^{out} = k$ and it is transitive if (i) all cycles are transitive and (ii) for any two nodes i and j in N, if i is connected to j then j is not connected to i. For simplicity, we assume that all positive claims are of equal value, normalized to 1.

2.2 Bankruptcy and shock diffusion

Define x_i as the total loss in external and interbank assets that bank i receives in case of a shock.

We use the standard defaulting rules in the literature (as in Eisenberg and Noe 2001, Elsinger et al. 2006 or Glasserman and Young 2015). Creditors have priority over shareholders and interbank liabilities are of equal priority. When a bank receives a shock, the losses on its external and interbank assets are reflected in capital loss. When its capital is depleted, the bank defaults. The condition of default of bank i is given by $x_i \geq w_i$. Then, the total loss-given-default that bank i impose on its creditors is

$$LGD^i = x_i - w_i \geq 0$$

A bankruptcy event is organized as follows: the defaulted bank liquidates all of its remaining assets and the liquidation proceeds are shared among creditors proportionally according to bank i's relative liabilities. We assume that for all assets, liquidating value is identical to book value, so that defaulted banks do not generate additional losses. Then, sharing liquidation proceeds is equivalent to share loss-given-default proportionally among creditors. Let's consider an example, depicted in Figure 3.

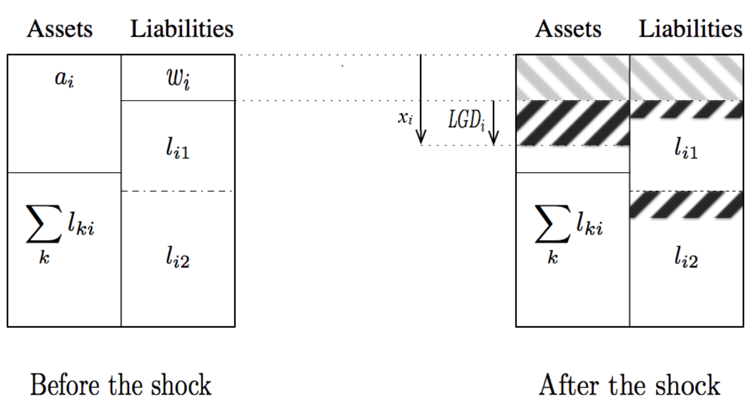


Figure 3: The shock and LGD

When bank i defaults from the external shock x_i, its liquidation proceeds are $a_i + \sum_k l_{ki} - x_i$. The loss-given-default that bank j suffers from the default of bank i is the difference between nominal liability and proportional repayment made by bank i to bank j.

$$\begin{aligned} LGD^i_j &= l_{ij} - (a_i + \sum_k l_{ki} - x_i)\frac{l_{ij}}{L_i} \\ &= \frac{l_{ij}}{L_i}\left[L_i - (a_i + \sum_k l_{ki})\right] + x_i\frac{l_{ij}}{L_i} \\ &= \frac{l_{ij}}{L_i}(-w_i) + \frac{l_{ij}}{L_i}x_i = LGD^i\frac{l_{ij}}{L_i} \end{aligned}$$

Thus, the shock is distributed proportionally according to relative liabilities. If the network is transitive, the shock diffuses in waves that do not come back to nodes who have been already affected by it.

3 Results

3.1 Limiting behavior of the shock

In order to compute the limit of losses in homogeneous, transitive networks as the number of banks gets large (when $n \to \infty$), we define regular networks of degree k and minimal transitive cycles of length c as follows.

Definition 1 *We say that a homogeneous, transitive network is a regular network with degree k and minimal transitive cycle of length $c \geq 3$ if (i) all nodes have in-degree and out-degree equal to k and (ii) starting from any bank $b \in N$ we can relabel the banks in a way such that for any $i \in N^{out}_b$*

$$N^{out}_i = \{i+1, i+c-1, i+c, i+c+1, ..., i+c+k-3\}$$

and

$$N^{in}_i = \{i-1, i-c+1, i-c, i-c-1, ..., i-c-k+3\}.$$

Figure 4 shows parts of (infinite) regular networks of degree $k = 2$ and minimal transitive cycle of length $c = 3$, $c = 4$, and $c = 5$, respectively. Each of the patterns shown below is assumed to be repeated infinitely because $n \to \infty$.

The term *minimal* transitive cycle of length c is used because a regular network, as defined previously, has many transitive cycles if $k > 2$. For example, if $k = c = 3$ and labeling the nodes as in the examples shown in Figure 4, we have that $\{12, 23, 13\} \subseteq g$ (transitive cycle of minimal length 3). Nevertheless, $\{12, 23, 34, 14\} \subseteq g$ is also a transitive cycle, but of length greater than 3.

We have the following result regarding the limit behavior of a single shock.

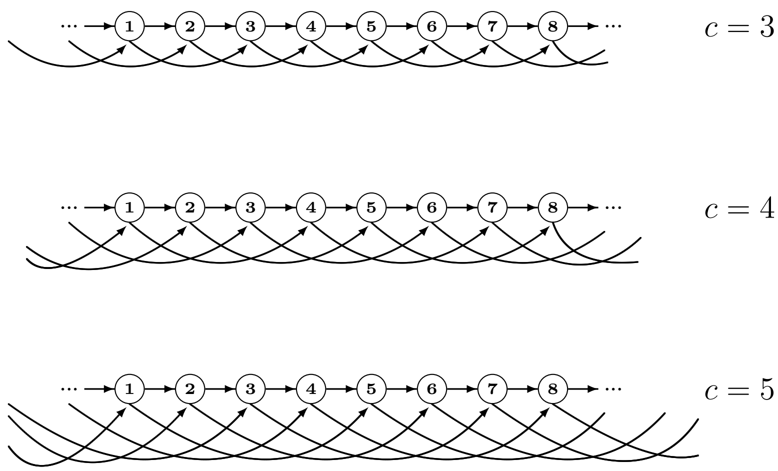


Figure 4: Regular networks of degree 2

Theorem 2 *Let w_i be equal to 0 for all $i \in N$ and assume one single external initial shock: there is one unique $j \in N$ such that (i) $x_j > 0$, and (ii) if $x_i > 0$ for $i \neq j$ then $i \in N_j^{out,\infty}$ and $x_i = \sum_{m \in N_i^{in}(g)} \frac{1}{k} x_m$. If the interdependency network of liabilities is a regular network of degree $k \geq 2$ and minimal cycle length $c \geq 3$ then for $i \in N_j^{out,\infty}$*

$$x_i \;\rightarrow\; \frac{2k}{2k + (k-1)(k+2c-6)} x_j \quad as \quad d(i,j) \;\rightarrow\; \infty$$

The proof is in the Appendix and it is built considering a natural relabeling/ordering of the nodes from their position/distance with respect to the node suffering the initial shock $j \in N$. We can then consider x_i for $i \in N_j^{out,\infty} = \{2, 3, 4, ...\}$ as an infinite sequence in $\Re_+$. This sequence is convergent in $\Re_+$ and its limit depends on x_j, k, and c as stated in Theorem 2. Figure 5 shows a numerical example of the behavior of the limit $\frac{x_i}{x_j}$ as c and k vary.

Theorem 2 shows that the losses received by banks that are connected to the node receiving the initial shock $j \in N$ do not go to zero even if banks are located infinitely far from j (as far as k and c are finite). A large value for the limit of the sequence x_i indicates that the structure itself facilitates the propagation of the losses without further consideration of other factors. Therefore we can consider the limit value of the losses as a measure of the contagiousness of the network.

3.2 Comparative statics

We discuss now how the limiting value of $\frac{x_i}{x_j}$, where j is the bank with the external, initial shock and $i \in N_j^{out,\infty}$ changes as k and/or c vary.

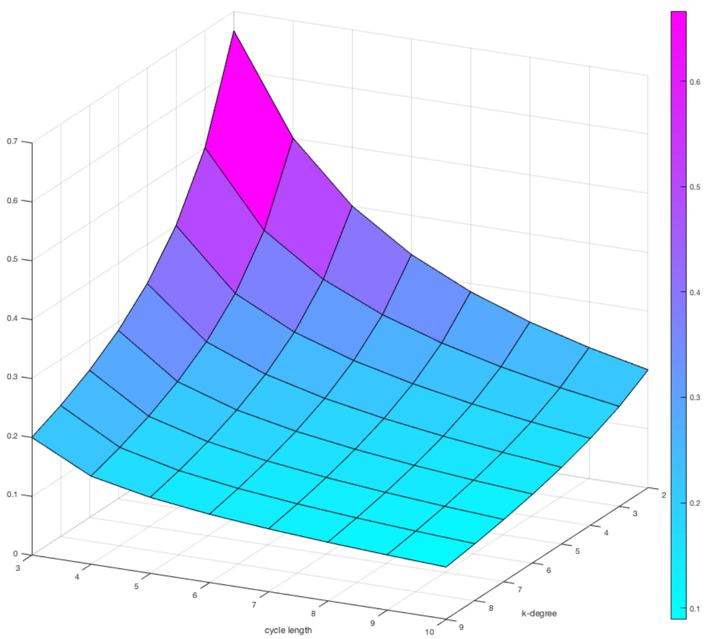
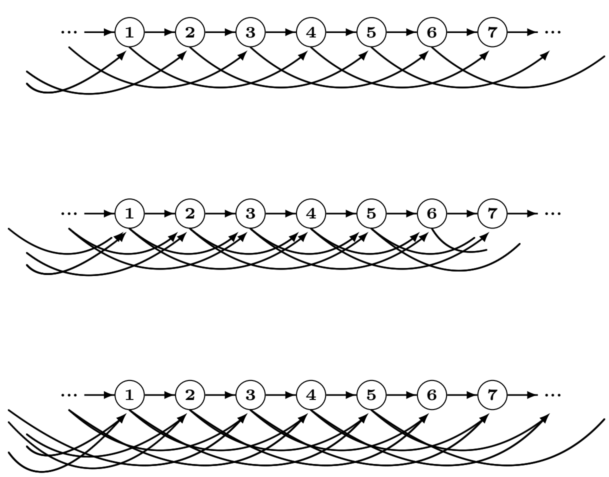


Figure 5: The limiting value of $\frac{x_i}{x_j}$ as $d(i,j)$ goes to infinity and j receives the unique initial, external shock, for $k = 2, ..., 9$ and $c = 3, ..., 10$

We observe from Theorem 2 that the limit of $\frac{x_i}{x_j}$ decreases with higher values of k or higher values of c (recall that $c \geq 3$). Therefore, according to Theorem 2, we can make two statements regarding the contagiousness of the network. First, increasing the length of minimal transitive cycles, while keeping the degree of connectivity constant, will make the network more robust, in the sense that it will dissipate a larger fraction of the shock during the diffusion process. Figure 4 provides an example of networks with degree equal to 2 but different lengths of minimal transitive cycles. Secondly, increasing the degree of connectivity, *while keeping the length of the minimal transitive cycle constant*, will also reduce the contagiousness of the network. Both of these effects can be observed in figure 5, as we move down along either one of the axis from any point.

Increasing the degree of connectivity might nevertheless decrease the length of the minimal transitive cycle. An example can be found in Figure 6 below. Starting from a regular network with $k = 2$ and $c = 4$, increasing the degree to $k = 3$ can be done in two different ways, such that the network remains regular as previously defined. First, we could add the link $i, i+2$ to the initial network, which would decrease the length of the minimal transitive cycle to 3. Secondly, we could also add the link $i, i+4$ to the initial network, which would keep the length of the minimal transitive cycle equal to 4. In general, to obtain a regular network of degree $k+1$ by adding one link per node to a regular network of degree k and minimal transitive cycle length c, there are two possible results. If we add the link $i, i+c-2$ for each $i \geq 1$ to the initial network (new *short* links) the length of the minimal transitive cycle decreases to $c-1$. If we add the link $i, i+k+c-2$ for each $i \geq 1$ to the initial network (new *long* links) the length of the minimal transitive cycle stays equal to c.



$k = 2,\ c = 4$

$k = 3,\ c = 3$

$k = 3,\ c = 4$

Figure 6: Increasing the degree of a regular network might decrease the length of the minimal transitive cycles

With regard to the addition of short links, we have the following proposition for the limit of losses, when the degree increases by one unit while the length of minimal transitive cycle decreases by one unit.

Proposition 3 *Let* $\bar{x}(k,c) = \frac{2k}{2k+(k-1)(k+2c-6)}$. *We have that* $\bar{x}(k+1,c-1) < \bar{x}(k,c)$ *if and only if* $c > 3 + \frac{k(k-1)}{2}$.

The proof of Proposition 3 is straightforward and therefore omitted. Proposition 3 states that there is a threshold for the length of the minimal transitive cycle such that the addition of a short link to each node reduces the contagiousness of the network.

Summing up, if the length of the minimal transitive cycle c is large enough, the contagiousness of the network is reduced if we consider an increase in degree, regardless of the type of additional links. When the length of the minimal transitive cycle is low, increasing the degree of connectivity has ambiguous results. If short links are added, the network becomes more contagious, while if long links are added then the network is less contagious.

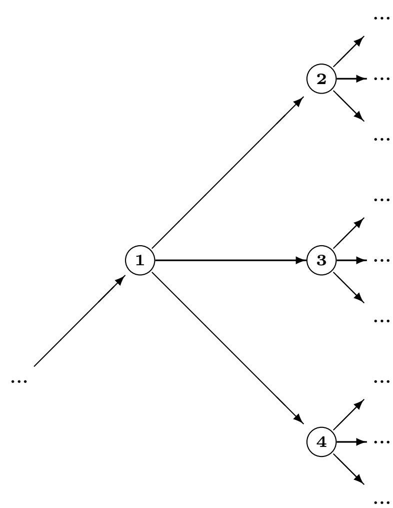
This result allows us to identify another factor that contributes to the non-monotonic relationship between density and systemic risk. Proposition 3 shows that increasing density may decrease or increase the extent of contagion depending on how the length of transitive cycles in the network changes as density varies.

4 Discussion

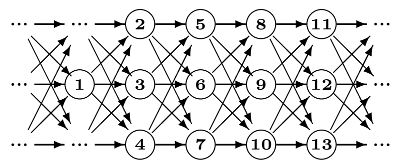
In this section, we extend our analysis of contagiousness and compare the regular networks with other families of networks that share some characteristics: the tree and the complete multipartite network. The families of networks that serve as benchmarks are all connected, transitive networks. This analysis will provide more insights to better understand the effect of the length of transitive cycles on the contagiousness of the network. Let us define the following two types of networks.

First, a connected, transitive network is considered to be a *tree of out-degree* k if (i) all nodes have out-degree equal to k and in-degree equal to 1, and (ii) for any two nodes i and j in N, if i is connected to j there is a *unique* path from j to i. Secondly, a connected, transitive, homogeneous network of degree k is a *complete multipartite network of degree* k if for any node $b \in N$ we can find (i) a set S_b of $k-1$ nodes such that for all $i \in S_b$ it holds that $N_i^{out} = N_b^{out}$, and (ii) a sequence of sets $\{S_b^t\}_{t=2,3,4,...}$ such that for all t and $i \in S_b^t$ it holds that $N_i^{out} = S_b^{t+1}$. Figure 7 shows an example of a tree of out-degree 3, a complete multipartite network of degree 3, and a regular network of degree 3 and minimal transitive cycle length equal to 3.

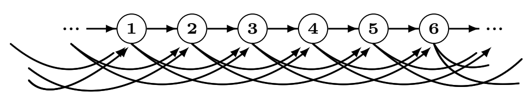
It is easy to see that in the case of the tree of out-degree equal to k the shock received by banks that are far from the source approaches zero when $w_i = 0$ for all $i \in N$. Recall that in a tree there will be a unique path connecting any $i \in N_j^{out,\infty}$ to j (the bank receiving the unique external shock). For any $i \in N_j^{out,\infty}$, each node in the path connecting i to



A tree



A complete multipartite network



A regular network

Figure 7: Networks with out-degree equal to 3

j diffuses $\frac{1}{k}$ of the shock received because $w_i = 0$ for all $i \in N$. Hence, $x_i = \frac{1}{k^{d(i,j)}} x_j$, where, recall, $d(i, j)$ is the distance from i to j (in this case the length of the unique path connecting them). As $d(i, j)$ tends to infinity for $i \in N_j^{out,\infty}$, we see that x_i tends to zero.

The case of the complete multipartite network of degree k is also easy to compute. The node receiving the initial external shock, j, diffuses $\frac{1}{k} x_j$ to each $i \in N_j^{out}$. Each $i \in N_j^{out}$ diffuses $\frac{1}{k^2} x_j$ to each $h \in N_i^{out}$. By definition of the complete multipartite network, each $h \in N_i^{out}$ is connected to all $i \in N_j^{out}$, hence receiving $x_h = \sum_{i \in N_j^{out}} \frac{1}{k^2} x_j = \frac{1}{k} x_j$. The shock received and transmitted by $i \in N_j^{out}$ is always equal to $\frac{1}{k} x_j$ and hence, as $d(i, j)$ tends to infinity for $i \in N_j^{out}$, x_i stays equal to $\frac{1}{k} x_j$.

These two types of networks, the tree and the complete multipartite one, illustrate well the role that in and out degrees have in the contagiousness properties of financial networks. If $k = 1$ both the tree and the complete multipartite network are equal to the infinite line $\{12, 23, 34, 45, 56, 67, ...\}$ (up to a relabelling of the nodes) and the shock received and transmitted by any $i \in N_j^{out,\infty}$ (j being the bank receiving the unique external, initial shock) is constant and equal to x_j. When $k \geq 2$ the tree and the complete multipartite network have a different shape which results in a different diffusion of the shock. In the tree, the shock received and transmitted by any $i \in N_j^{out,\infty}$ is decreasing exponentially until it reaches zero because the out-degree being greater than the in-degree helps spread the shock, making it smaller as it travels further through the network. In the complete multipartite network the in-degree and the out-degree are equal. This creates the possibility of connecting banks in $N_j^{out,\infty}$ to j through many different paths.[4] This multiplicity of paths prevents the shock to decrease to zero as it gets further away from j because there is *accumulation without amplification* through the multiple paths connecting the nodes.

This distinct behavior of shock diffusion in these two networks can also be related to the neighborhood growth in Morris (2000). In the tree, the bank receiving the initial external shock has k out-neighbors. Each of these k out-neighbors have k distinct out-neighbors, the initial external shock has an effect over k^2 *new* nodes after two iterations of the set of out-neighborhood. We note that after l iterations of the set of out-neighborhood k^l nodes are *newly* added. In the complete multipartite network, the bank receiving the initial external shock also has k out-neighbors, but each of these k out-neighbors have the same k out-neighbors. After l iterations of the set of out-neighborhood we still find k new banks being affected by the initial external shock in the multipartite network. Morris (2000) shows that in social coordination games (coordination games played on a network) new behaviors are *potentially* more contagious in networks where there is slow neighborhood growth, which means that the number of new out-neighbors at each iteration of the set of out-neighborhood does not grow exponentially. The diffusion behavior of the shock is consistent with this view. The tree is less contagious because the shock goes to zero as we get far from the initial shock in our analysis and the neighborhood growth in the sense of Morris (2000) is exponential. The complete multipartite network is very contagious because the shock does not go to zero as we get far from the initial shock in our analysis and the neighborhood growth in the sense of Morris is not exponential (it is constant).

What happens in the case of the regular network? It is also true that the neighborhood growth is constant given the regularity of the network: after the node $j + c - 1$ is reached, there are always $k + c - 3$ new out-neighbors added at each iteration step. It might be tempting to assume that the regular network is less contagious than the complete multipartite network by looking at the neighborhood growth, as $c \geq 3$. We have the following proposition comparing the two limiting values of the shock as we get far from the bank receiving the initial shock.

Proposition 4 *Recall that* $\bar{x}(k,c) = \frac{2k}{2k+(k-1)(k+2c-6)}$. *We have that* $\bar{x}(k,c) < \frac{1}{k}$ *if and only if* $c > 3 + \frac{k}{2}$.

The proof of Proposition 4 is straightforward and therefore omitted. Proposition 4 states that there is a threshold for the length of the minimal transitive cycle such that

[4] This multiplicity of paths does not imply the existence of cycles in the network because links are directed.

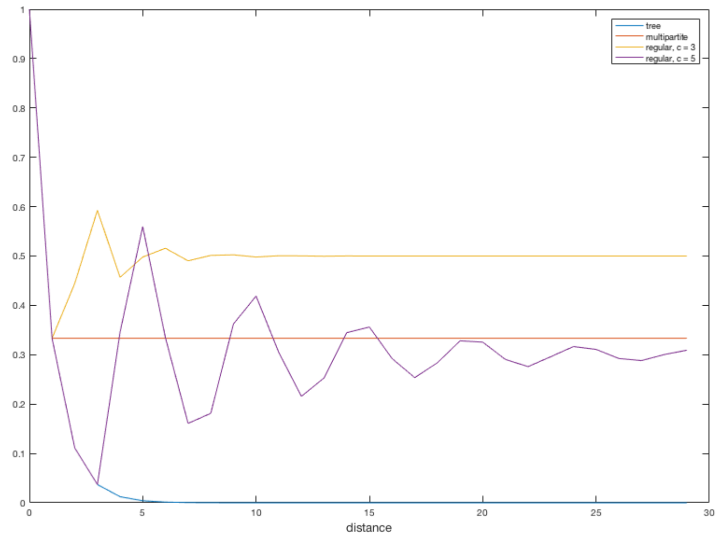


Figure 8: The value of $\frac{x_i}{x_j}$ as a function of the distance $d(i,j)$ to bank j receiving the unique initial, external shock, for networks of degree equal to 3

a regular network can be less contagious than a complete multipartite network. For an illustration, Figure 8 shows that with the same degree of 3, the regular network with $c = 5$ is less contagious than the multipartite network, while the regular network with $c = 3$ is more contagious.

In particular, if $c = 3$ the regular network will be more contagious than the complete multipartite network for any value of $k > 1$. As c approaches infinity the shape of the regular network approaches the one of the tree. We also note that the value of the threshold increases with the degree of the network. If the network gets denser (in the sense of higher in and out degree) the minimal transitive cycle length has to be greater too so that the regular network is less contagious than the complete multipartite network of the same degree. This result demonstrates another important role of minimal transitive cycles. Networks with very similar patterns and characteristics can have different behaviors regarding shock diffusion, depending on the value of the length of minimal transitive cycles.

5 Concluding comments

Our analysis provides new insights about shock diffusion in financial networks by focusing on the role of minimal transitive cycles. Using large regular networks, where all nodes have equal in-degree and out-degree and with the same pattern of links repeating infinitely, we allow an initial shock to diffuse as a flow into the system. The contagiousness of a network is measured by the limit of the losses of banks that are located at an infinite distance to the first defaulted bank. This measure captures how a structure of liabilities may or may not facilitate the propagation of losses in spite of other external/financial factors.

Our analysis allows variations of the length of the minimal transitive cycle as far as the number of financial institutions tends to infinity. We find that contagiousness is decreasing in the length of the minimal transitive cycle. Increasing the degree has ambiguous effects, depending on whether the length of minimal transitive cycle decreases or not after the addition of new links. Finally, similar network structures can have different level of contagiousness when the length of minimal transitive cycles is above or below a certain threshold.

Our results contribute to the literature (see first paragraph of introduction) in which the relationship between density (or diversity) on the network of financial intermediaries and the extent of contagion is found to be non-monotonic and to depend on external factors. We complement previous results by Allen et al (2012), who showed that clustering in the financial network might entail higher systemic risk. We believe that an indicator capturing the length of transitive cycles in a network can be easily included in the design of capital requirement recommendations by central banks.

Further work includes applying numerical methods to compute how the extent of contagion in more realistic financial networks depends on the length of transitive or intransitive cycles[5].

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Appendix

Proof of Theorem 2

Let us fix $i = 1$ to be the institution receiving the unique external shock. Given the transitivity nature of our network, only nodes in $N_1^{out,\infty}$ can potentially receive a shock from their in-neighbors. Given the regularity of our network, we can now label the nodes following the natural order defined by the network. Formally, the labeling satisfies that (i) $N_1^{out,\infty} = \{2, 3, 4, 5, ...\}$, and (2) for every i and j in $N_1^{out,\infty}$: $i < j$ if and only if $j \in N_i^{out,\infty}$. The regularity of the network and the transitivity requirements guarantee that the labeling makes sense. The examples shown in Figure 4 are an illustration of such a natural labeling of the nodes.

We make use of the following Lemma.

Lemma. Let (N, g) be a regular network of degree k and minimal cycle length c. Assume $w_i = 0$ for all $i \in N$. We fix $i = 1$ as the label for the node that receives the unique external shock. Starting from $i = 1$ we consider a labeling of nodes as explained above. Recall that x_i denotes total loss in assets that bank i receives in case of a shock (coming from the external asset or from interbank assets). We have that if $c = 3$ then

$$\frac{1}{k}x_1 + \frac{2}{k}x_2 + ... + \frac{k-1}{k}x_{k-1} + x_k = x_1,$$

while if $c \geq 4$ then

$$\frac{1}{k}x_1 + \frac{2}{k}x_2 + ... + \frac{k-1}{k}x_{k-1} + \frac{k-1}{k}\left(x_k + ... + x_{k+c-4}\right) + x_{k+c-3} = x_1.$$

Proof of Lemma. We consider first the case when $c = 3$. Recall that $w_i = 0$ for all $i \in N$. Hence node k receives a fraction $\frac{1}{k}x_j$ from each $j \in N_k^{in}$. By definition of the network and the labeling of the nodes the only nodes $j \in N_k^{in}$ such that $x_j > 0$ are the ones in the set $\{1, ..., k-1\}$. Hence,

$$x_k = \frac{1}{k}\sum_{j=1}^{k-1} x_j.$$

Substituting x_k we obtain

$$\frac{1}{k}x_1 + \frac{2}{k}x_2 + ... + \frac{k-1}{k}x_{k-1} + x_k = \frac{2}{k}x_1 + \frac{3}{k}x_2 + ... + \frac{k-1}{k}x_{k-2} + x_{k-1}.$$

We proceed to substitute x_{k-1}. Following the same argument as before,

$$x_{k-1} = \frac{1}{k}\sum_{j=1}^{k-2} x_j.$$

Substituting x_{k-1} we obtain

$$\frac{1}{k}x_1 + \frac{2}{k}x_2 + ... + \frac{k-1}{k}x_{k-1} + x_k = \frac{3}{k}x_1 + \frac{4}{k}x_2 + ... + \frac{k-1}{k}x_{k-3} + x_{k-2}.$$

Applying the argument recursively, we arrive to

$$\frac{1}{k}x_1 + \frac{2}{k}x_2 + ... + \frac{k-1}{k}x_{k-1} + x_k = \frac{k-1}{k}x_1 + x_2.$$

Given that $x_2 = \frac{1}{k}x_1$ we obtain, by substituting x_2, that

$$\frac{1}{k}x_1 + \frac{2}{k}x_2 + ... + \frac{k-1}{k}x_{k-1} + x_k = x_1.$$

We consider now the case when $c \geq 4$. We apply a similar argument as before. Recall that $w_i = 0$ for all $i \in N$. Hence node $k+c-3$ receives a fraction $\frac{1}{k}x_j$ from each $j \in N^{in}_{k+c-3}$. We note that, by definition of the network and the labelling of the nodes, the only nodes $j \in N^{in}_{k+c-3}$ such that $x_j > 0$ are the ones in the set $\{1, ..., k-2, k+c-4\}$. Hence,

$$x_{k+c-3} = \frac{1}{k}\sum_{j=1}^{k-2} x_j + \frac{1}{k}x_{k+c-4}.$$

Substituting x_{k+c-3} we obtain

$$\begin{gathered}\tfrac{1}{k}x_1 + \tfrac{2}{k}x_2 + ... + \tfrac{k-1}{k}x_{k-1} + \tfrac{k-1}{k}\left(x_k + ... + x_{k+c-4}\right) + x_{k+c-3} = \\ \tfrac{2}{k}x_1 + \tfrac{3}{k}x_2 + ... + \tfrac{k-1}{k}x_{k-2} + \tfrac{k-1}{k}x_{k-1} + \tfrac{k-1}{k}\left(x_k + ... + x_{k+c-5}\right) + x_{k+c-4}.\end{gathered}$$

We proceed to substitute x_{k+c-4}. Following the same argument as before,

$$x_{k+c-4} = \frac{1}{k}\sum_{j=1}^{k-3} x_j + \frac{1}{k}x_{k+c-5}.$$

Substituting x_{k+c-4} we obtain

$$\frac{1}{k}x_1+\frac{2}{k}x_2+...+\frac{k-1}{k}x_{k-1}+\frac{k-1}{k}\left(x_k+...+x_{k+c-4}\right)+x_{k+c-3}=$$
$$\frac{3}{k}x_1+\frac{4}{k}x_2+...\frac{k-2}{k}x_{k-3}+\frac{k-1}{k}\left(x_{k-2}+x_{k-1}+x_k+...+x_{k+c-6}\right)+x_{k+c-5}.$$

Applying the argument recursively for $j\geq c$, noting that $c\in N_1^{out}$ but $i\notin N_1^{out}$ for $2<i\leq c-1$, we arrive to

$$\frac{1}{k}x_1+\frac{2}{k}x_2+...+\frac{k-1}{k}x_{k-1}+x_k=\frac{k-1}{k}\left(x_1+...+x_{c-2}\right)+x_{c-1}.$$

Given that $x_i=\frac{1}{k}x_{i-1}$ for $1<i\leq c-1$ we obtain, by substituting recursively, that

$$\frac{1}{k}x_1+\frac{2}{k}x_2+...+\frac{k-1}{k}x_{k-1}+\frac{k-1}{k}\left(x_k+...+x_{k+c-4}\right)+x_{k+c-3}=x_1.$$

This completes the proof of the Lemma. $\square$

We proceed now to prove the statement of Theorem 2. Recall that we have labeled the nodes such that (i) $i=1$ is the node receiving the unique external shock, (ii) $N_1^{out,\infty}=\{2,3,4,5,....\}$, and (iii) for every i and j in $N_1^{out,\infty}$: $i<j$ if and only if $j\in N_i^{out,\infty}$.

We can rewrite the sequence $x_1,x_2,x_3,...$ in matrix form as

$$x^{[i+1]}=Ax^{[i]}$$

with $x^{[i+1]}=\begin{pmatrix}x_{i+1}\\x_{i+2}\\\vdots\\x_{i+k+c-3}\end{pmatrix}$, $x^{[i]}=\begin{pmatrix}x_i\\x_i\\\vdots\\x_{i+k+c-4}\end{pmatrix}$ and

$$A_{(k+c-3\times k+c-3)}=\begin{pmatrix}0&1&0&\cdots&0&0&\cdots&0&0\\0&0&1&\cdots&0&0&\cdots&0&0\\0&0&0&\cdots&1&0&\cdots&0&0\\0&0&0&\cdots&0&1&\cdots&0&0\\\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\0&0&0&\cdots&0&0&\cdots&1&0\\0&0&0&\cdots&0&0&\cdots&0&1\\\frac{1}{k}&\frac{1}{k}&\frac{1}{k}&\cdots&\frac{1}{k}&0&\cdots&0&\frac{1}{k}\end{pmatrix}.$$

In the last row of matrix A we find the first $k-1$ elements and the last element to be equal to $\frac{1}{k}$ (so k elements are equal to $\frac{1}{k}$) and the rest of elements to be equal to 0. It is easy to see that

$$x^{[n]}=A^nx^{[1]}\tag{1}$$

with $x^{[1]} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{k+c-3} \end{pmatrix}$.

Given that A is a row stochastic matrix we have that 1 is a simple eigenvalue of A and that the spectral radius of A is equal to 1. We also know that A is irreducible and primitive.[6] Hence, by equation (8.3.10) in Meyer (2000), p. 674, we have that

$$\underset{n\to\infty}{Lim}\, A^n = \frac{r.l^T}{l^T.r} \tag{2}$$

where r and l are, respectively, the right and left eigenvectors corresponding to the eigenvalue 1, l^T is the transpose of l (l is written as a column vector, so l^T is a row vector).

Given that A is row stochastic, the right eigenvector is equal to the vector of ones. To compute the left eigenvector we solve

$$(l_1, ..., l_{k+c-3}) \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \\ \frac{1}{k} & \frac{1}{k} & \frac{1}{k} & \cdots & \frac{1}{k} & 0 & \cdots & 0 & \frac{1}{k} \end{pmatrix} = (l_1, ..., l_{k+c-3}),$$

and obtain

$$\begin{gathered} l_i = \tfrac{i}{k}, \text{ for } 1 \leq i \leq k-1 \\ l_i = \tfrac{k-1}{k}, \text{ for } k-1 < i \leq k+c-4 \\ l_{k+c-3} = 1. \end{gathered}$$

Substituting in (2) to compute the limit of A^n we obtain

$$\underset{n\to\infty}{Lim}\, A^n = \frac{1}{\sum_{i=1}^{k+c-3} l_i} \begin{pmatrix} l_1 & l_2 & \cdots & l_{k+c-3} \\ l_1 & l_2 & \cdots & l_{k+c-3} \\ \cdots & \cdots & \cdots & \cdots \\ l_1 & l_2 & \cdots & l_{k+c-3} \end{pmatrix}$$

[6]A nonnegative $n \times n$ matrix A is irreducible if and only if the graph $G(A)$, defined to be the directed graph on nodes $1, 2, ..., n$ in which there is a directed edge leading from i to j if and only if $a_{ij} > 0$, is strongly connected (see Meyer 2000, p. 671). A nonnegative $n \times n$ matrix A is primitive if it is irreducible and at least one diagonal element is positive, i.e., the trace of the matrix is positive (see Meyer 2000, p. 678). Furthermore, we also know that A^{k+2c-6} is a positive matrix.

Hence,

$$\underset{n\to\infty}{Lim}\,(x_n) = \frac{1}{\sum_{i=1}^{k+c-3} l_i} \sum_{i=1}^{k+c-3} l_i x_i$$

Note that

$$\sum_{i=1}^{k+c-3} l_i x_i = \begin{cases} \frac{1}{k}x_1 + \frac{2}{k}x_2 + ... + \frac{k-1}{k}x_{k-1} + x_k, & \text{if } c = 3 \\ \frac{1}{k}x_1 + \frac{2}{k}x_2 + ... + \frac{k-1}{k}x_{k-1} + \frac{k-1}{k}\left(x_k + ... + x_{k+c-4}\right) + x_{k+c-3} & \text{if } c \geq 4. \end{cases}$$

Hence, by Lemma,

$$\underset{n\to\infty}{Lim}\,(x_n) = \frac{1}{\sum_{i=1}^{k+c-3} l_i} x_1 = \frac{2k}{2k + (k-1)(k+2c-6)} x_1.$$

This completes the proof of Theorem 2. $\square$